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The minimal conformal $O(N)$ vector sigma model at $d = 3$

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Abstract

For the minimal $O(N)$ sigma model, which is defined to be generated by the $O(N)$ scalar auxiliary field alone, all n -point functions, till order $1/N$ included, can be expressed by elementary functions without logarithms. Consequently, the conformal composite fields of m auxiliary fields possess at the same order such dimensions, which are m times the dimension of the auxiliary field plus the order of differentiation.

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1. Introduction

The conformal $O(N)$ vector sigma model has been studied in a series of basic works in [1–4] and then been analysed from several points of view in [5–7]. Recently it has attracted interest as a candidate for an $\text{AdS}_4/\text{CFT}_3$ correspondence [8], where the AdS_4 theory is a special higher spin gauge field theory on AdS space of the type investigated in a large series of papers by Vasiliev [9–12], see also [13, 14].

We want to describe here the properties of the ‘minimal’ interacting model, consisting only of the ‘auxiliary’ or ‘Lagrange multiplier’ field α , which is a scalar under Lorentz and internal $O(N)$ transformations. Thus we are interested in the set of n -point functions $\langle \alpha(x_1) \cdots \alpha(x_n) \rangle$, where we consider the ‘physical’ spacetime dimension $d = 3$, because in the references given above, d is a parameter in the open interval $d \in (2, 4)$. The n -point functions are given by a $1/N$ -expansion. We shall describe the simplifications arising by the restrictions to $d = 3$.

The $\text{AdS}_4/\text{CFT}_3$ correspondence mentioned above is in a speculative state and ought to be tested in detail. There are at present two arguments supporting it:

1. The three-point function of the auxiliary field in the conformal sigma model and the three-point function of the scalar field in the (minimal) higher spin field theory on AdS [15] vanish (see equation (33)).

2. The holographic images of the higher spin gauge fields are supposed to be symmetric tensor currents, both of even rank. The conformal partial wave expansion of the four-point function of the α field at perturbative order $O(1/N)$ contains besides the operators, which are composed of α fields, only conserved currents [21]. Moreover, as we will show in this work, at $d = 3$ there are only contributions of α composites at order 1.

In this paper we collect material which is relevant for detailed studies of this correspondence. However, it may be also of general interest. Namely, the n -point functions of the α field at $d = 3$ can be given explicitly in terms of simple analytic functions, if one restricts to perturbative order $O(1/N)$. Since this is a very exceptional property of an interacting field theory it is justified to call it ‘ $O(1/N)$ exact solvability’.

Besides the interacting conformal model a free theory based solely on the free $O(N)$ vector field $\phi = (\phi_1, \dots, \phi_n)$, which is also a scalar under Lorentz transformations, enables us to calculate the n -point functions of the ‘scalar current’

$$J(x) = \sum_{j=0}^N \phi_j(x) \phi_j(x). \quad (1)$$

If we perturb the free $O(N)$ model by the ‘double-trace’ operator

$$\frac{\lambda}{2N} J(x)^2 \quad (2)$$

the free theory flows from the corresponding unstable ultraviolet fixed point to the stable infrared fixed point [16, 17], which belongs to the interacting $O(N)$ theory mentioned above. During this flow, the field dimension of J changes from its free field value $\Delta_- = 1$ to the field dimension $\Delta_+ = 2$ of α . Both fixed points are connected by the AdS/CFT correspondence to the same bulk field with mass $m^2 = -2$, therefore we have

$$\Delta_{\pm} = \mu \pm \sqrt{\mu^2 - 2} \quad (3)$$

where we introduced the convenient abbreviation $\mu = d/2$, which will be used throughout the paper. The theories at both fixed points are connected by a Legendre transformation [18], see section 3.

The paper is organized as follows: In section 2, we quickly resume some important facts about the $O(N)$ sigma model and how its infrared fixed point can be described. Section 3 is devoted to the study of the ultraviolet fixed point and it is recapitulated, how this fixed point is connected with the IR fixed point by a Legendre transform. In section 4, we cite the result of [7] about the α four-point function at order $O(1/N)$, and show that a hitherto unevaluated part vanishes at $d = 3$ as well as the logarithmic contributions. This results in a four-point function, which can be expressed in terms of elementary functions. In section 5, we briefly comment on the appearance of anomalous dimensions at $d = 3$.

2. The $O(N)$ model at its interacting fixed point

A perturbative expansion of a CFT is formulated by a skeleton graph expansion, where the propagators and the three-point vertices are essentially fixed by conformal covariance. In the $O(N)$ sigma model at the infrared stable fixed point we thus have the full propagators for the auxiliary field α and the $O(N)$ vector ϕ

$$\begin{aligned} G(x_{12}) &:= \langle \alpha(x_1) \alpha(x_2) \rangle = (x_{12}^2)^{-\beta} && \text{denoted by a dashed line} \\ D(x_{12}) &:= \langle \phi_i(x_1) \phi_j(x_2) \rangle = \delta_{ij} (x_{12}^2)^{-\delta} && \text{denoted by a solid line.} \end{aligned} \quad (4)$$

The latter must be taken into account in the internal lines of the skeleton graphs for the n -point functions of α . We mention that the normalization constants of the two-point functions are absorbed in the coupling constant z of the interaction vertex, which is given by

$$z^{\frac{1}{2}} \int dx \alpha(x) \sum_{j=1}^N \phi_j(x)^2. \tag{5}$$

z assumes a fixed value at the interacting fixed point (critical value), which has the expansion

$$z = \sum_{i=1}^{\infty} \frac{z_i}{N^i}. \tag{6}$$

The field dimensions β and δ decompose into a canonical and an anomalous part η ,

$$\beta = 2 + \eta(\alpha) \tag{7}$$

$$\delta = \mu - 1 + \eta(\phi) \tag{8}$$

where the anomalous parts can be expanded in $1/N$:

$$\eta(\alpha) = \sum_{i=1}^{\infty} \frac{\eta_i(\alpha)}{N^i} \tag{9}$$

$$\eta(\phi) = \sum_{i=1}^{\infty} \frac{\eta_i(\phi)}{N^i}. \tag{10}$$

The expansion coefficients $\eta_1(\phi)$, $\eta_2(\phi)$, $\eta_3(\phi)$ and $\eta_1(\alpha)$ are known [2–4, 19].

In general, an integral

$$\int dy \prod_{i=1}^M ((y - x_i)^2)^{-\gamma_i} \tag{11}$$

is conformally invariant if the dimensions γ_i satisfy the constraint of ‘uniqueness’

$$\sum_{i=1}^M \gamma_i = d. \tag{12}$$

However, the interaction vertex (5) has to be UV renormalized, thereby violating uniqueness, i.e. $\beta + 2\delta = d - 2\kappa$ implies $\eta(\alpha) = -2\eta(\phi) - 2\kappa$, where the deviation from uniqueness κ can be expanded in a series in $1/N$, similar to η .

Let us explain how the coupling constants z and the anomalous dimension $\eta(\phi)$ can be obtained from the conformal bootstrap equations

$$D^{-1} + \text{[diagram: tadpole with dashed line]} + \text{[diagram: triangle with dashed lines]} + \dots = 0 \tag{13}$$

$$\frac{2}{N} G^{-1} + \text{[diagram: bubble]} + \text{[diagram: diamond with vertical lines]} + \dots = 0 \tag{14}$$

which are equations for the respective amputated two-point functions. The amputations are performed by the inverses of the propagators (4). Explicitly, the propagators are kernels of the form

$$F(x) = (x^2)^{-\lambda} \tag{15}$$

and the respective inverses F^{-1} are defined by

$$\int dx_2 F(x_{12})F^{-1}(x_{23}) = \delta(x_{13}). \quad (16)$$

Then one easily gets

$$F^{-1}(x) = p(\lambda)(x^2)^{-(d-\lambda)} \quad (17)$$

with

$$p(\lambda) = \pi^{-2\mu} \frac{a(\lambda - \mu)}{a(\lambda)} \quad a(\lambda) = \frac{\Gamma(\mu - \lambda)}{\Gamma(\lambda)}. \quad (18)$$

Now we insert (4) and (5) into the bootstrap equations (13) and (14), restrict to the leading terms, respectively, and obtain

$$p(\delta) = -z \quad p(\beta) = -\frac{1}{2}Nz. \quad (19)$$

The second equation in (19) implies

$$z_1 = -2p(2) = 2\pi^{-2\mu} \frac{(\mu - 2)\Gamma(2\mu - 2)}{\Gamma(\mu)\Gamma(1 - \mu)} \quad (20)$$

whereas the first equation in (19) gives

$$\eta_1(\phi) = 2 \frac{\sin \pi \mu}{\pi} \frac{\Gamma(2\mu - 2)}{\Gamma(\mu + 1)\Gamma(\mu - 2)}. \quad (21)$$

At $d = 3$ we furthermore obtain

$$\eta_1(\phi)|_{d=3} = \frac{4}{3} \quad \eta_1(\alpha)|_{d=3} = -\frac{32}{3} \quad \kappa_1|_{d=3} = 4. \quad (22)$$

3. Free field theory and a Legendre transform

We consider the free field theory of the composite fields (normal ordering understood)

$$J(x) = \sum_{i=1}^N \phi_i(x)\phi_i(x). \quad (23)$$

With the normalization of the ϕ field (4) we obtain for the two-, three- and four-point function of J by Wick's theorem

$$\langle J(x_1)J(x_2) \rangle = 2N(x_{12}^2)^{-d+2} \quad (24)$$

$$\langle J(x_1)J(x_2)J(x_3) \rangle = 8N(x_{12}^2 x_{23}^2 x_{13}^2)^{-(\mu-1)} \quad (25)$$

and

$$\begin{aligned} \langle J(x_1)J(x_2)J(x_3)J(x_4) \rangle = & 4N^2[(x_{12}^2 x_{34}^2)^{-d+2} + (x_{13}^2 x_{24}^2)^{-d+2} + (x_{14}^2 x_{23}^2)^{-d+2}] \\ & + 16N[(x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2)^{-\mu+1} + (x_{12}^2 x_{24}^2 x_{43}^2 x_{31}^2)^{-\mu+1} + (x_{13}^2 x_{32}^2 x_{24}^2 x_{41}^2)^{-\mu+1}]. \end{aligned} \quad (26)$$

From [8] we have learned that this theory is connected to the interacting sigma model by a Legendre transformation with respect to the dual field α ,¹

$$\int dx J(x)\alpha(x). \quad (27)$$

¹ In contrast to the previous section, we now choose to set the coupling constant of the three vertex to unity, which implies that the normalization of the two-point function is the one given in (17).

Thus let us perform this Legendre transformation. To this end, we build up every diagram of α fields that can be constructed from the n -point functions of J by attaching α propagators, normalized as

$$\langle \alpha(x_1)\alpha(x_2) \rangle = K(x_{12})^{-2} \tag{28}$$

to the respective legs. Since α is dual to J , the respective two-point functions are inverses to each other

$$\int dx_3 (x_{13}^2)^{-d+2} (x_{32}^2)^{-2} = p(2)^{-1} \delta(x_{12}). \tag{29}$$

Consequently, we get the α two-point function from the Legendre transformed J two-point function

$$\int dx_3 dx_4 \langle \alpha(x_1)\alpha(x_3) \rangle \langle J(x_3)J(x_4) \rangle \langle \alpha(x_4)\alpha(x_2) \rangle = 2NK^2 p(2)^{-1} (x_{12}^2)^{-2} \tag{30}$$

if we use the normalization constant (see (17))

$$K = \frac{1}{2} \left(\frac{z_1}{N} \right)^{\frac{1}{2}}. \tag{31}$$

Performing the Legendre transformation on the three-point function of J (25) gives the α three-point function

$$\langle \alpha(x_1)\alpha(x_2)\alpha(x_3) \rangle = \begin{array}{c} 1 \\ | \\ \triangle \\ / \quad \backslash \\ 2 \quad \quad 3 \end{array} \tag{32}$$

which can be integrated to give

$$\langle \alpha(x_1)\alpha(x_2)\alpha(x_3) \rangle = N \left(\frac{z_1}{N} \right)^{\frac{3}{2}} v(2, \mu - 1, \mu - 1)^2 v(2, 1, 2\mu - 3) (x_{12}^2 x_{13}^2 x_{23}^2)^{-1} \tag{33}$$

where

$$v(\alpha_1, \alpha_2, \alpha_3) = \pi^\mu \prod_{i=1}^3 \frac{\Gamma(\mu - \alpha_i)}{\Gamma(\alpha_i)}. \tag{34}$$

Now we observe that $v(2, 1, 2\mu - 3)$ in (33) contains a factor $\Gamma(2\mu - 3)$ in the denominator, thus the α three-point function vanishes at $d = 3$.

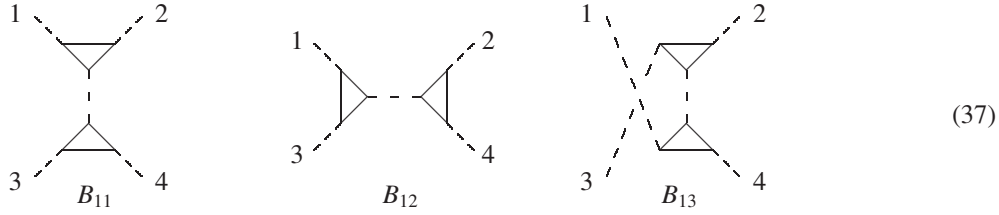
The Legendre transform of the J four-point function gives the α four-point function, which consists of three disconnected graphs

$$\begin{array}{ccc} 1 \text{ --- } 2 & \begin{array}{c} 1 | 2 \\ | | \\ 3 | 4 \end{array} & \begin{array}{c} 1 \diagdown 2 \\ \diagup \diagdown \\ 3 \diagup 4 \end{array} \\ A_1 & A_2 & A_3 \end{array} \tag{35}$$

three box graphs

$$\begin{array}{ccc} \begin{array}{c} 1 \diagdown 2 \\ | | \\ 3 \diagup 4 \end{array} & \begin{array}{c} 1 \diagdown 2 \\ | | \\ 3 \diagup 4 \end{array} & \begin{array}{c} 1 \diagdown 2 \\ | | \\ 3 \diagup 4 \end{array} \\ B_{21} & B_{22} & B_{23} \end{array} \tag{36}$$

which will be treated in detail in the next section and the one-particle reducible graphs



which have been evaluated in [7] and are quoted in the appendix. At $d = 3$, these graphs have a second-order zero, which seem to originate from each α three-point function contained in the graph.

4. The four-point function in the interacting conformal sigma model and its $d = 3$ limit

Now we consider the sigma model at its interacting critical point. This means we have to calculate the four-point function $\langle \alpha(x_1)\alpha(x_2)\alpha(x_3)\alpha(x_4) \rangle$ up to order $1/N$, which consists of the three types of graphs mentioned in the previous section: the disconnected graphs, the one-particle reducible graphs, which vanish at $d = 3$ and the box graphs, which are investigated in this section. In [7], they are only partly computed and, as far as we know, have never been fully evaluated. Let us repeat the results of this reference: The box graphs B_{2j} , $j = 1, 2, 3$, have the structure

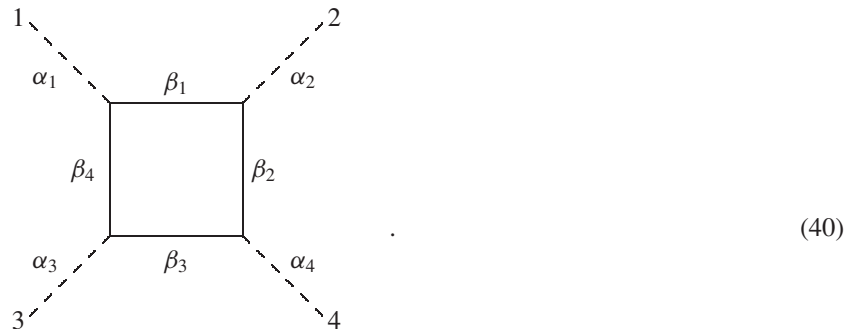
$$B_{2j} = (x_{12}^2 x_{34}^2)^{-2} \sum_{m,n \geq 0} \frac{u^n (1-v)^m}{n! m!} [-a_{nm}^{(2j)} \log u + b_{nm}^{(2j)} + u^{\mu-3} c_{nm}^{(2j)}] \quad (38)$$

see equations (5.11)–(5.13) of [7], where the coefficients $a_{nm}^{(2j)}$ are given in equations (C.7), (C.9) and (C.11) in [7]. The conformally invariant variables u, v are defined by

$$u = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \quad (39)$$

We observe that all of them contain a factor $\Gamma(2\mu - 4)^{-1}$, giving a zero at $d = 3$. The coefficients $c_{nm}^{(2j)}$, $j = 1, 2$ (equations (C.8) and (C.10)²) have even a double zero from the $\Gamma(2\mu - 3)^{-2}$, but this zero is compensated by a double pole for all but a few (n, m) . These surviving terms lead to current exchanges, as discussed in [21]. We shall return to them below. Moreover, we also have $c_{nm}^{(23)} = 0$.

The coefficients $b_{nm}^{(2j)}$, $j = 1, 2, 3$, are not completely evaluated, but they all have a zero at $d = 3$. To see this, let us turn to the general box graph (40), which has been discussed in a more general framework in [5].



² In this equation there is a wrong factor of $1/n!$, which should be removed.

The integration technique consisted in doing two unique three-vertex integrals at opposite positions in the graph, performing one of the two remaining four-vertex integrals and transforming the second into a Barnes type integral via Symanzik’s technique [20]. This results in

$$(x_{13}^2)^{-\alpha_1} (x_{34}^2)^{\alpha_1 - \alpha_3} (x_{24}^2)^{-\alpha_2} B(u, v) \tag{41}$$

with

$$B(u, v) = \sum_{m,n=0}^{\infty} \frac{u^n (1-v)^m}{n!m!} [u^{\alpha_1} c_{nm}^{(1)} + u^{\alpha_2} c_{nm}^{(2)} + u^{\beta_3} c_{nm}^{(3)}]. \tag{42}$$

The non-evaluated part of this graph is a contribution to $c_{nm}^{(2)}$ (see equation (A.9) in [5])

$$\begin{aligned} c_{nm,1}^{(2)} = & \pi^{2\mu} \frac{v(\alpha_1, \beta_1, \beta_4)v(\alpha_4, \beta_2, \beta_3)}{\prod_{j=1}^4 \Gamma(\delta_j)} (2\pi i)^{-2} \int_{-i\infty}^{i\infty} dx \int_{-i\infty}^{i\infty} dy \Gamma(-x)\Gamma(-y) \\ & \times \Gamma(\delta_4 + x + y)\Gamma(\mu - \delta_1 + x + y)\Gamma(\delta_1 + \delta_2 - \mu - y)\Gamma(\delta_1 + \delta_3 - \mu - x) \\ & \times a(\gamma_1)a(\gamma_3)a(\gamma_2 + \gamma_4) \frac{(\gamma_4)_n(\mu - \gamma_3)_n(\gamma_2)_{n+m}(\mu - \gamma_1)_{n+m}}{(\gamma_2 + \gamma_4)_{2n+m}(\gamma_2 + \gamma_4 - \mu + 1)_n} \end{aligned} \tag{43}$$

where

$$\delta_1 = \mu - \beta_1 \quad \delta_2 = 2\mu - \alpha_1 - \alpha_2 \quad \delta_3 = \alpha_3 \quad \delta_4 = \mu - \beta_2 \tag{44}$$

$$\gamma_1 = \mu - \alpha_4 - y \quad \gamma_2 = \alpha_2 \quad \gamma_3 = \beta_1 + x + y \quad \gamma_4 = \mu - \beta_3 - x. \tag{45}$$

The crucial factor is $\Gamma(\delta_2)^{-1}$, since at the end we set $\alpha_1 = \alpha_2 = 2$. This produces a simple zero at $d = 3$. The question is whether this zero is cancelled by a pole.

If

$$\Delta := \alpha_1 - \alpha_2 \tag{46}$$

tends to zero, then poles in Δ in $c_{nm}^{(1)}$ and $c_{nm}^{(2)}$ (42) arise, which cancel each other and give

$$u^{\alpha_1} [-a_{nm} \log u + b_{nm}]. \tag{47}$$

These poles are obviously independent of the pole at $2\mu - \alpha_1 - \alpha_2 = 2\mu - 4 = -1$ and can be neglected in this context. The standard way of evaluating the integral (43) is by shifting the contours to $+\infty$ and summing up the residues. This results in generalized hypergeometric series of argument 1, whose poles in the parameters are difficult to evaluate.

Thus we used the method of ‘contour pinches’, which is explained in the appendix. There we also show that there is no pole cancelling the zero $\Gamma(\delta_2)^{-1}$ and we conclude that there are no contributions from the non-evaluated integrals to the α four-point function at $d = 3$.

Therefore, the only contributions to the α four-point function arise from the coefficients $c_{nm}^{(2j)}$, $j = 1, 2$, which have been extracted for general d in [7]. Applying this result for $d = 3$ gives for the connected part

$$\langle \alpha(x_1) \cdots \alpha(x_4) \rangle |_{\text{conn}, d=3} = (x_{12}^2 x_{34}^2)^{-2} \frac{1}{N} [-u^{-\frac{3}{2}}(1 + u - v) + (uv)^{-\frac{3}{2}}(1 - u - v)]. \tag{48}$$

However, this expression is not crossing symmetric, i.e. invariant under the replacements

$$(a) \quad 1 \leftrightarrow 2 \quad u \leftrightarrow v \tag{49}$$

$$(b) \quad 2 \leftrightarrow 3 \quad u \mapsto \frac{1}{u} \quad v \mapsto \frac{v}{u}. \tag{50}$$

The unique scalar field $(n\alpha)_{0,0}$ has anomalous dimension

$$\delta((n\alpha)_{0,0}) = 2n - \frac{4}{N}\eta_1(\phi)\frac{2\mu - 1}{\mu - 2} \left[n(n - 2)(\mu - 1) + \binom{n}{2}(2\mu - 1)(\mu - 2) \right] + O(1/N^2) \tag{56}$$

see equation (3.33) in [22]. It is remarkable that for $d = 3$ the quadratic dependence of the anomalous dimension on n reduces to a linear one, and that we get the special case $l = t = 0$ of (55).

For $l \neq 0, t = 0$ and arbitrary n we must expect that several different conformal fields with the same quantum numbers l, t, n have different anomalous dimensions. We can use equations (3.28)–(3.42) and (3.17) of [22] to check the case $d = 3$. We find indeed (55) with the degeneracy lifted presumably at $O(1/N^2)$. For quasi-free fields there is a cohomology approach to the construction of different conformal fields [23] which can be also applied to this case.

Appendix

Consider the example

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \Gamma(\alpha + s)\Gamma(\beta + s)\Gamma(\gamma - s)\Gamma(\delta - s) = \frac{\Gamma(\alpha + \gamma)\Gamma(\beta + \gamma)\Gamma(\alpha + \delta)\Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)}. \tag{A1}$$

The integrand has two sequences of poles

$$\gamma - s = -N_1 \quad \delta - s = -N_2 \quad (N_{1,2} \in \mathbb{N}_0) \tag{A2}$$

tending to $+\infty$, and two sequences of poles

$$\alpha + s = -M_1 \quad \beta + s = -M_2 \quad (M_{1,2} \in \mathbb{N}_0) \tag{A3}$$

tending to $-\infty$ in the s -plane. The integral contour is chosen in such a way that all poles from the ‘left sequence’ (A3) are to the left of the contour, and all poles from the ‘right sequence’ (A2) are to its right. In the case that such a choice is not possible, namely if a left and a right series intersect in a (necessarily finite) number of points, the contour is said to be pinched in these points and poles arise there. This pinching occurs in the following cases:

$$s = \gamma + N_1 = -\alpha - M_1 \quad \text{i.e. if } \alpha + \gamma = -N_1 - M_1 \tag{A4}$$

$$s = \gamma + N_1 = -\beta - M_2 \quad \text{i.e. if } \beta + \gamma = -N_1 - M_2 \tag{A5}$$

$$s = \delta + N_2 = -\alpha - M_1 \quad \text{i.e. if } \alpha + \delta = -N_2 - M_1 \tag{A6}$$

$$s = \delta + N_2 = -\beta - M_2 \quad \text{i.e. if } \beta + \delta = -N_2 - M_2. \tag{A7}$$

Thus, we conclude that the possible poles are just those of the numerator of the result in (A1), i.e. the poles of

$$\Gamma(\alpha + \gamma)\Gamma(\beta + \gamma)\Gamma(\alpha + \delta)\Gamma(\beta + \delta) \tag{A8}$$

and that the integrand is equal to this meromorphic function times an arbitrary holomorphic function, as (A1) shows.

Now we treat $c_{nm,1}^{(2)}$ from equation (43) in this way. The relevant integral is

$$\frac{1}{(2\pi i)^2} \int dx \int dy \Gamma \left[\begin{matrix} -x, -y, \alpha_4 + n + m + y, \beta_3 - \alpha_2 - n + x, \beta_4 - \alpha_4 - y \\ \mu - \alpha_4 - y, \mu + \alpha_2 - \beta_3 + 2n + m - x \end{matrix} \right] \times \Gamma(\alpha_3 - \beta_1 - x, \mu - \beta_2 + x + y, \mu - \beta_1 + n - x - y)(\mu - \beta_3 - x)_n. \tag{A9}$$

Note that the Γ -function has no zeros. With

$$a = \mu - \beta_1 + n \quad b = \beta_3 - \alpha_2 - n \quad c = \mu - \beta_2 \quad d = \alpha_3 - \beta_1 \quad (\text{A10})$$

we get the right sequences of poles

$$-x = -M_1 \quad a - x - y = -M_2 \quad d - x = -M_3 \quad (M_{1,2,3} \in \mathbb{N}_0) \quad (\text{A11})$$

and the left sequences

$$b + x = -N_1 \quad c + x + y = -N_2 \quad (N_{1,2} \in \mathbb{N}_0) \quad (\text{A12})$$

which combine into six pinch sequences of x

$$\begin{aligned} b = -N_1 - M_1 & \quad c + y = -N_2 - M_1 & \quad a + b - y = -N_1 - M_2 \\ a + c = -N_2 - M_2 & \quad b + d = -N_1 - M_3 & \quad c + d + y = -N_2 - M_3 \end{aligned} \quad (\text{A13})$$

leading to

$$\Gamma(b, a + c, b + d, c + y, c + d + y, a + b - y). \quad (\text{A14})$$

Next we consider the y -pinches of

$$\Gamma(-y, e + y, f - y, c + y, c + d + y, a + b - y) \quad (\text{A15})$$

where

$$e = \alpha_4 + n + m \quad f = \beta_4 - \alpha_4. \quad (\text{A16})$$

From this we read the right sequences of poles

$$-y = -R_1 \quad f - y = -R_2 \quad a + b - y = -R_3 \quad (R_{1,2,3} \in \mathbb{N}_0) \quad (\text{A17})$$

and the left sequences

$$e + y = -S_1 \quad c + y = -S_2 \quad c + d + y = -S_3 \quad (S_{1,2,3} \in \mathbb{N}_0). \quad (\text{A18})$$

Then we obtain pinches in the following cases:

$$\begin{aligned} e = -S_1 - R_1 & \quad e + f = -S_1 - R_2 & \quad e + a + b = -S_1 - R_3 \\ c = -S_2 - R_1 & \quad c + f = -S_2 - R_2 & \quad c + a + b = -S_2 - R_3 \\ c + d = -S_3 - R_1 & \quad c + d + f = -S_3 - R_2 & \quad c + d + a + b = -S_3 - R_3 \end{aligned} \quad (\text{A19})$$

with the corresponding meromorphic factor

$$\Gamma(b, c, e, a + c, b + d, c + d, c + f, e + f, c + d + f, c + a + b) \Gamma(e + a + b, a + b + c + d). \quad (\text{A20})$$

This does not include a pole at $\mu = 3/2$, which would be necessary to cancel the simple zero in the prefactor $\Gamma(\delta_2)^{-1}$ in (43).

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